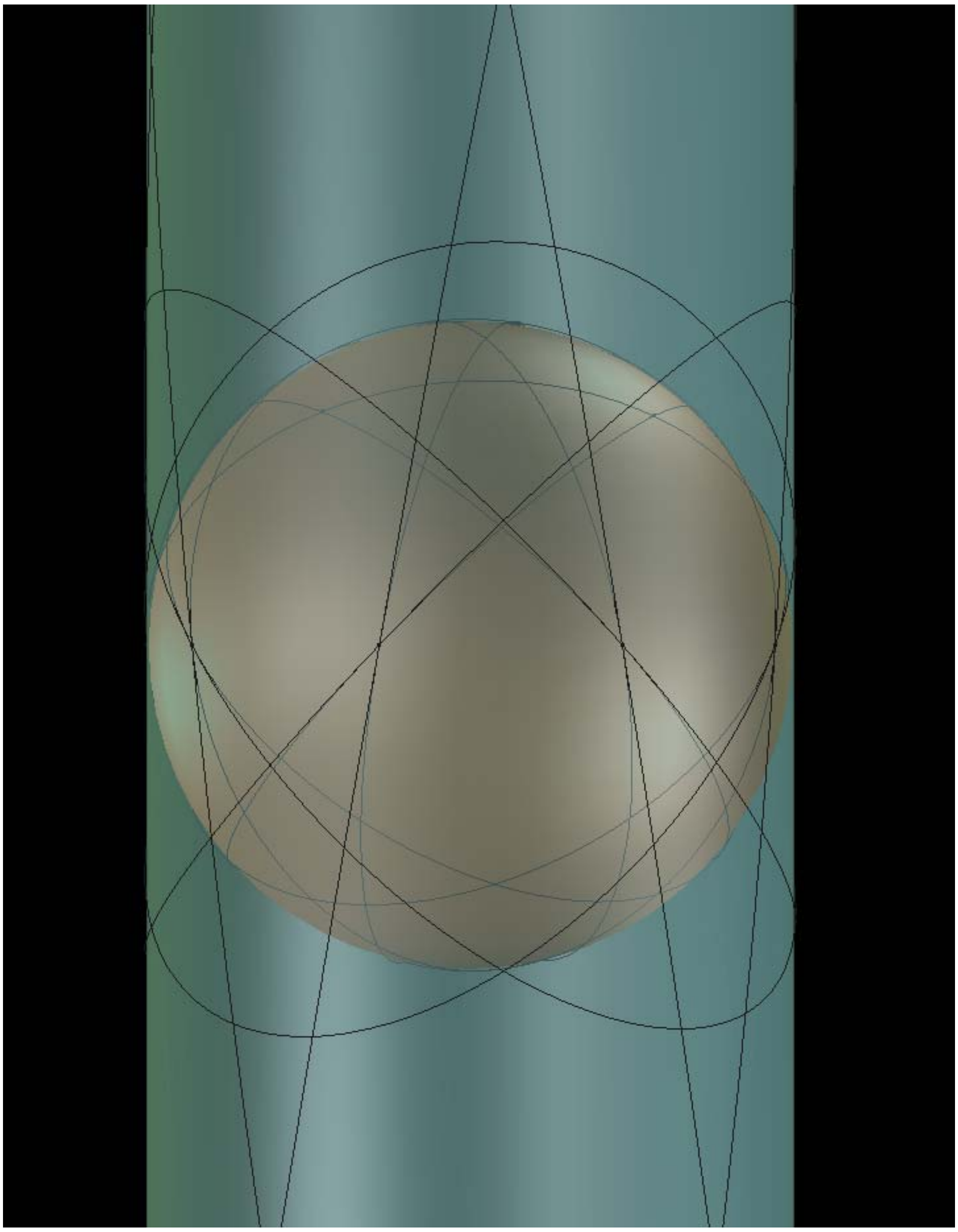
Author: Dirk Bertels **Hobart, January, 2005**

Intersections of great circles on a sphere



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Introduction

This writing is the 'verbal' companion to a series of animation programs that illustrate the mathematics and practical outcome of a particular type of dome.

This particular dome was inspired by Buckminster[1].
To construct this dome in the virtual world, we need to go through the following steps:

Top part of the dome

- Construct a sphere.
- Position a pentagon, lying horizontally and close to the top of the sphere, intersecting it at about $9/10^{th}$ of the sphere's radius[2].
- Position another pentagon, also lying horizontally but underneath the first one, intersecting the sphere at about $6/10^{th}$ of the sphere's radius.
- For one pentagon, draw the great circles formed by each successive pair of its vertices.
- Do the same for the other pentagon.
- Find out where the great circles intersect the 'equator' of the sphere.
- Find out where the great circles intersect each other.

Lower part of the dome

- Project the great circles onto a vertical cylinder that encompasses the sphere, touching it. The great circles now become ellipses.
- Find the intersections of those 'great ellipses'.
- Find intersections of these great ellipses and some horizontal circles forming the base and cutting at points where the above intersections occurred.

Total dome

- Take the top hemisphere of the great circles and divide it into equal distances.
- Take the lower part of the 'great ellipses' and bound it to a chosen value. Divide the elliptical curves into equal distances.
- Join them.

As you can see, there's some geometry involved, and even some calculus at the end, where we are required to calculate lengths along the sphere and cylinder. But overall the maths is quite basic, in fact, that's the beauty of this project, it forces you to review fundamental topics in mathematics, such as vectors and the polar coordinate system.

At the end of this document are some stills taken from 2 animations I developed using the theory described here. The first animation just illustrates the sphere with its great circles

[1] See Bamboo Dome illustration at back of this document.
[2] Thanks to a friend of mine, Robert Oates, who alerted me to this system of using 2 pentagons to construct the dome.

and the cylinder with its great ellipses. This combination reveals a mathematical symmetry that is aesthetically quite attractive (assuming that's the kind of thing that turns you on…).

The second animation illustrates the actual physical domes.

Knowing all too well that reading demands much more concentration than writing (just as listening is much harder than talking), I try to keep the writing simple and clear, making a gradual transition from topic to topic. However, some topics are worth exploring and some extra attention may be given where it is not strictly needed.

Any constructive comments, critique, questions will be welcomed. Email me at dbertels@utas.edu.au, or use the forum at my website www.ids.org/~dbertels

Read the following chapter carefully as it clarifies some fundamental definitions used in this paper.

Definitions

Great Circles

Great circles are circles drawn on the surface of a sphere, the only condition being that the center of each of these great circles coincides with the sphere's center. Therefore the radii of all these great circles are the same as the radius of the sphere they encompass.

Any two points on a sphere that are not antipodal (not positioned in a straight line with the center) uniquely define a great circle.

Great circles are a universal (natural) phenomenon since the shortest distance between 2 points on a sphere is always a section of a great circle. This is why great circles are extensively used in fields such as astronomy, aviation, surveying, etc…

Any two great circles intersect exactly twice.

Any three non-collinear points (not lying on a straight line) in space define a plane.

The two points defining a great circle, together with the origin of the sphere can be used to uniquely identify a plane that cuts the sphere through its center. **The great circle is actually the intersection of this plane with the sphere**.

Coordinate systems

The back page of this paper illustrates an example of a "ZYX" coordinate system. However, while the orientation of the X, Y, and Z-axes are standard now in the mathematics field, in more practical fields they often differ to suit the application they are used for. The supplement shows the orientation and location of the axes used in this paper. This particular configuration is chosen because it is the standard used by openGL, the graphics library for the C++ programming language used for the animations.

Knowing the orientation of the 3D world you work in is fundamental. It is useful to get used to the idea of 'up' as 'Y', 'right' as 'X', and 'forward' as 'Z'.

Equations and descriptions found in mathematics literature often need converting to suit the orientation of the world. . In the "Conversion to spherical coordinates" chapter, I included the equations for a different coordinate system to demonstrate this. Generally all that needs doing is replace the axes' names (X, Y, and Z) with the ones used in the application.

Vectors[3]

Assuming the origin of the sphere is known we can represent each point on the sphere by a vector.

Given points P_1 and center O[4], vector $\mathbf{OP_1}$ has coordinates $\langle x_1, y_1, z_1 \rangle$

Cross product

The cross product of 2 vectors is the multiplication of 2 vectors whose origins are at the center. The result is a third vector perpendicular to the other 2. This perpendicular vector uniquely identifies the plane (in 3D) where the original vectors reside in.[5]

Given the vectors $\mathbf{OP_1} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{OP_2} = \langle x_2, y_2, z_2 \rangle$

The cross product of these two vectors result in a third vector,

$$OP_1 \times OP_2 = \langle y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2 \rangle$$

Likewise for vectors $\mathbf{OP_3} = \langle x_3, y_3, z_3 \rangle$ and $\mathbf{OP_4} = \langle x_4, y_4, z_4 \rangle$

$$OP_3 \times OP_4 = \langle y_3 z_4 - z_3 y_4, z_3 x_4 - x_3 z_4, x_3 y_4 - y_3 x_4 \rangle$$

For ease of notation, let

$$a = y_1 z_2 - z_1 y_2 \qquad d = y_3 z_4 - z_3 y_4$$

$$b = z_1 x_2 - x_1 z_2 \qquad e = z_3 x_4 - x_3 z_4$$

$$c = x_1 y_2 - y_1 x_2 \qquad f = x_3 y_4 - y_3 x_4$$

We can simplify the above to

$$OP_1 \times OP_2 = \langle a, b, c \rangle$$

$$OP_3 \times OP_4 = \langle d, e, f \rangle$$

[3] Refer to my document on vectors under the maths folder. This chapter contains some extracts from this document.

[4] Note the difference in notation: A point in 3D is represented by the set of coordinates (x, y, z), while the vector pointing from (0, 0, 0) to the point (x, y, z) is represented by <x, y, z>.

[5] The cross-product is used in animation to determine the angle of all the surfaces so that lighting can be applied proportionally. The technique is called 'normalisation'.

Note: The length of the vector obtained by the cross product of OP_1 and OP_2 is

$$\left|OP_1 \times OP_2\right| = \left|OP_1\right| * \left|OP_2\right| * \sin\theta$$

Where θ is the angle between OP_1 and OP_2 in the range $0 < \theta < \pi$

The dot product

The dot product of 2 vectors multiplies the corresponding coordinates and adds the values.

For vectors $OA_1 = \langle x_1, y_1, z_1 \rangle$ and $OB = \langle x_2, y_2, z_2 \rangle$

$$OA \cdot OB = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Note that the result of the dot product is a real number (a scalar) and that its operation symbol is a 'dot'. Contrast this with the cross product that results in another vector and whose operation symbol is a 'cross'.

The dot product is used to determine the angle between 2 vectors.

$$OA \cdot OB = |OA||OB|\cos\theta$$

$$\cos\theta = \frac{OA \cdot OB}{|OA||OB|}$$

In fact, **we can think of the dot product as measuring the extent to which the 2 vectors are pointing in the same direction**.

If OA and OB point in the same general direction,

$$OA \cdot OB > 0$$

If OA and OB point in the same general opposite direction,

$$OA \cdot OB < 0$$

If OA and OB point to exactly the opposite direction,

$\theta = \pi \rightarrow \cos \pi = -1 \rightarrow OA \cdot OB = -|OA||OB|$

If OA and OB are perpendicular (orthogonal),

$\theta = \pi/2 \rightarrow \cos \pi/2 = 0 \rightarrow OA \cdot OB = 0$

It is this last property of the dot product that will prove to be useful to us.

Equation of a great circle

Recall from the introduction that any 2 points on a sphere that are not antipodal uniquely identify a great circle on that sphere.
To construct the equation that is valid for all the possible points on a great circle we go through the following steps:

1. Draw the vectors from the center of the sphere to the 2 points:

$$OP_1 = \langle x_1, y_1, z_1 \rangle$$

$$OP_2 = \langle x_2, y_2, z_2 \rangle$$

2. Draw the vector which is perpendicular to both these vectors[6]:

$$OP_1 \times OP_2 = \langle a, b, c \rangle$$

We can now define the great circle by stating that

If the point lies somewhere on the surface of the sphere and
If the vector formed from the center of the sphere to this point ***is perpendicular to the perpendicular*** *of the 2 original vectors,* then *this point is located on the great circle defined by the 2 original vectors.*

The 'perpendicular to the perpendicular' ensures that the arc formed by the 2 initial points on the surface of the sphere rotates around the center point in right angles to the perpendicular.

In mathematical speak, if $OP_x = \langle x, y, z \rangle$ is a vector pointing from the origin of the sphere to a point on the surface of that sphere, then according to the standard equation for a sphere[7]

$$x^2 + y^2 + z^2 = r^2 \qquad (1)$$

And if 2 vectors $OP_1 = \langle x_1, y_1, z_1 \rangle$ and $OP_2 = \langle x_2, y_2, z_2 \rangle$ are given to identify a unique great circle, then the cross product of these vectors produce the vector perpendicular to OP_1 and OP_2

$$OP_1 \times OP_2 = \langle a, b, c \rangle$$

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[6] See the "Cross product" chapter to determine a, b, and c.

[7] In the supplement "Spherical equations", r is represented by 'ρ' to distinguish it from 'r' in 2D

If the dot product of this perpendicular vector with OP_x proves to be zero, then the point must be located on a plane cutting the sphere.[8]:

$$(OP_1 \times OP_2) \cdot OP_x = 0 \quad \rightarrow \quad \langle a, b, c \rangle \cdot \langle x, y, z \rangle = 0 \quad \rightarrow \quad ax + by + cz = 0 \qquad (2)$$

It is the intersection with this plane and the sphere that produces a great circle. In other words, we need to solve equations (1) and (2) for x, y and z.

In order to plot the points of a great circle, we can increment the **y** values by a fixed amount and use a procedure to determine the other coordinates. This way, **y** becomes a known variable. To distinguish from the unknown variables, we symbolise the unknowns with capital letters.

from (2)

$$aX + by + cZ = 0$$

$$Z = \frac{-aX - by}{c} \qquad (3)$$

from (1)

$$X^2 = 1 - y^2 - Z^2 \qquad (4)$$

(3) in (4) gives

$$X^2 = 1 - y^2 - \left(\frac{-aX - by}{c} \right)^2$$

$$X^2 = 1 - y^2 - \left(\frac{a^2 X^2 + 2abyX + b^2 y^2}{c^2} \right)$$

$$c^2 X^2 - c^2 + c^2 y^2 = -a^2 X^2 - 2abyX - b^2 y^2$$

Rearrange to fit a polynomial equation…

$$\left(c^2 + a^2\right) X^2 + \left(2aby\right) X + (b^2 y^2 + c^2 y^2 - c^2) = 0 \qquad (5)$$

[8] If the coordinates of the sphere were not taken in consideration, the equation of the great circle would become the equation of the plane.

Let …

$$u = c^2 + a^2$$

$$v = 2aby$$

$$w = b^2 y^2 + c^2 y^2 - c^2$$

… then (5) can be simplified to

$$uX^2 + vX + w = 0$$

with roots

$$x = \frac{-v \pm \sqrt{v^2 - 4uw}}{2u}$$

This of course enables you to find z, using (1)

$$z = \pm\sqrt{1 - y^2 - x^2}$$

Intersection of great circles with the equator

(Special case for Y = 0)

Gives coordinates for Great Circle at the equator of the sphere

$$Z = \frac{-aX}{c}$$

$$X^2 = 1 - Z^2$$

$$X^2 = 1 - \left(\frac{-aX}{c}\right)^2$$

$$X^2 = 1 - \left(\frac{a^2X^2}{c^2}\right)$$

$$\frac{a^2X^2}{c^2} = 1 + X^2$$

$$a^2X^2 - c^2X^2 = c^2$$

$$X^2\left(a^2 - c^2\right) = c^2$$

$$X^2 = \frac{c^2}{a^2 - c^2}$$

$$X = \pm\sqrt{\frac{c^2}{a^2 - c^2}}$$

First intersection at

$$\sqrt{\frac{c^2}{a^2 - c^2}}, 0, \frac{-a\sqrt{\frac{c^2}{a^2 - c^2}}}{c}$$

Second intersection at

$$-\sqrt{\frac{c^2}{a^2-c^2}},0,\frac{a\sqrt{\frac{c^2}{a^2-c^2}}}{c}$$

You will note that the great circles of the pentagon1 and pentagon2 cross each other at this equator if both pentagons have the same longitudinal orientation.

Equation of the intersection of 2 great circles

We use exactly the same procedure to define the second great circle, using the 2 vectors

$OP_3 = \langle x_3, y_3, z_3 \rangle$ and $OP_4 = \langle x_4, y_4, z_4 \rangle$

$$(OP_3 \times OP_4) \cdot OP_x = 0 \quad \rightarrow \quad \langle d, e, f \rangle \cdot \langle x, y, z \rangle = 0 \quad \rightarrow \quad dx + ey + fz = 0 \qquad (6)$$

It follows that if equations (1), (2), and (3) hold true, then the point $\mathbf{P_x}$ is located on the sphere and on both great circles, that is, we defined the necessary conditions for this point to exist. We also know from the introduction that exactly 2 such points must exist for any 2 great circles.

Solving the equations

All that remains now is solving equations (1), (2), and (6) for x, y, and z

from (2)

$$x = \frac{(-by - cz)}{a} \qquad (7)$$

Substitute (7) in (6)

$$dx + ey + fz = 0$$
$$d\frac{(-by - cz)}{a} + ey + fz = 0$$
$$-\frac{dby}{a} - \frac{dcz}{a} + ey + fz = 0$$
$$ey - \frac{dby}{a} = \frac{dcz}{a} - fz$$
$$y\left(e - \frac{db}{a}\right) = z\left(\frac{dc}{a} - f\right)$$
$$y\left(\frac{ea - db}{a}\right) = z\left(\frac{dc - fa}{a}\right)$$
$$y = z\frac{dc - fa}{ea - db}$$

let $h = \frac{dc - fa}{ea - db}$ then $y = hz$ (8)

Substitute (8) in (7) and isolate z

$$x = \frac{(-bhz - cz)}{a} = z\frac{-bh - c}{a}$$

let $g = \frac{-bh - c}{a}$ then $x = gz$ (9)

Substitute (9), (8), and (7) into (1)

$$x^2 + y^2 + z^2 = r^2$$
$$(gz)^2 + (hz)^2 + z^2 = r^2$$
$$z^2(g^2 + h^2 + 1) = r^2$$
$$z = \pm\sqrt{\frac{r^2}{g^2 + h^2 + 1}}$$

let $k = \sqrt{\frac{r^2}{g^2 + h^2 + 1}}$

The two great circles intersect at the points[9]:

$$(gk, hk, k), (-gk, -hk, -k)$$

[9] As stated before, remember that 'r' which is a variable in 'k' is represented by 'ρ' in the supplement.

Conversion to spherical coordinates

If you're like me and hopelessly confused about geographical terminology, here's a refresher:

Meridian

Great circle passing through both poles.

Latitude

The latitude of a point on earth is its angular distance from the equator measured upon the curved surface of the earth.
Referred to as 'ϕ' in the "Spherical Coordinates" appendix. It is the angle formed between the point and the Y-axis.

Longitude

The longitude of a point on earth is the angular distance from a standard meridian (usually through Greenwich) to the meridian running through this point.
Referred to as 'θ' in the "Spherical Coordinates" appendix. It is the angle formed between the point and the YZ-plane.

$P_1 = (lat_1, lon_1)$

$P_2 = (lat_2, lon_2)$

In the "Y Points up, X points right, Z points forward" coordinate system (as used in the supplement):

$$lat_1 = \cos^{-1}\left(\frac{hk}{r}\right)$$

$$lon_1 = \tan^{-1}\left(\frac{gk}{k}\right)$$

$$lat_2 = \cos^{-1}\left(\frac{-hk}{r}\right)$$

$$lon_2 = \tan^{-1}\left(\frac{-gk}{-k}\right)$$

In the "Z points up, Y points right, X points forward" coordinate system[10]:

$$lat_1 = \cos^{-1}\left(\frac{k}{r}\right) \qquad lon_1 = \tan^{-1}\frac{hk}{gk}$$

$$lat_2 = \cos^{-1}\left(\frac{-k}{r}\right) \qquad lon_2 = \tan^{-1}\frac{-hk}{-gk}$$

Recapitulation

$$OP_1 = \langle x_1, y_1, z_1 \rangle$$

$$OP_2 = \langle x_2, y_2, z_2 \rangle$$

$$OP_3 = \langle x_3, y_3, z_3 \rangle$$

$$OP_4 = \langle x_4, y_4, z_4 \rangle$$

$$a = y_1 z_2 - z_1 y_2 \qquad d = y_3 z_4 - z_3 y_4$$

$$b = z_1 x_2 - x_1 z_2 \qquad e = z_3 x_4 - x_3 z_4$$

$$c = x_1 y_2 - y_1 x_2 \qquad f = x_3 y_4 - y_3 x_4$$

$$g = \frac{-bh - c}{a} \qquad h = \frac{dc - fa}{ea - db}$$

$$k = \sqrt{\frac{r^2}{g^2 + h^2 + 1}}$$

The two great circles intersect at the points:

$$(gk, hk, k), (-gk, -hk, -k)$$

[10] Used in standard mathematical text books, here from James Stewart's "Calculus" ISBN 0-534-35949-3

Example

Assume the circle has its center at (0, 0, 0)
To ensure we are working with the same circle, all we need to do is to make sure that the value for 'r' remains the same.

For example take the arbitrary coordinate values -2, 3, and 2.
Switching the values and polarity of the coordinates ensures that 'r' remains the same. For the coordinates given above,

$$r^2 = x^2 + y^2 + z^2 = (-2)^2 + 3^2 + 2^2 = 3^2 + (-2)^2 + 2^2 = 17$$

$$r = \sqrt{17}$$

First great circle

Using the values given above,

$$OP_1 = \langle x_1, y_1, z_1 \rangle = \langle -2,3,2 \rangle$$

$$OP_2 = \langle x_2, y_2, z_2 \rangle = \langle 3,-2,2 \rangle$$

Determine the cross-product (orthogonal vector)

$$a = y_1 z_2 - z_1 y_2 = 6 + 4 = 10$$

$$b = z_1 x_2 - x_1 z_2 = 6 + 4 = 10$$

$$c = x_1 y_2 - y_1 x_2 = 4 - 9 = -5$$

$$OP_1 \times OP_2 = \langle a, b, c \rangle = \langle 10,10,-5 \rangle$$

Equation of the plane containing the first great circle

$$ax + by + cz = 0$$

$$10x + 10y - 5z = 0$$

Second great circle

To facilitate testing the equations, we use one point of the first great circle plus a new one that also adheres to the same radius. In other words, $OP_1 = OP_3$ which should be showing up as one of the intersection points.

$$OP_3 = \langle x_3, y_3, z_3 \rangle = \langle -2,3,2 \rangle$$

$$OP_4 = \langle x_4, y_4, z_4 \rangle = \langle -2,2,3 \rangle$$

$$d = y_3 z_4 - z_3 y_4 = 9 - 4 = 5$$

$$e = z_3 x_4 - x_3 z_4 = -4 + 6 = 2$$

$$f = x_3 y_4 - y_3 x_4 = -4 + 6 = 2$$

$$OP_3 \times OP_4 = \langle d, e, f \rangle = \langle 5,2,2 \rangle$$

$$dx + ey + fz = 0$$

$$5x + 2y + 2z = 0$$

Point of intersections

$$h = \frac{dc - fa}{ea - db} = \frac{-25 - 20}{20 - 50} = \frac{3}{2}$$

$$g = \frac{-bh - c}{a} = \frac{-15 + 5}{10} = -1$$

$$k = \sqrt{\frac{r^2}{g^2 + h^2 + 1}} = \sqrt{\frac{17}{\left(\frac{3}{2}\right)^2 + (-1)^2 + 1}} = 2$$

Therefore the points of intersection between the 2 great circles are

$$(gk, hk, k), (-gk, -hk, -k) = (-2,3,2), (2,-3,-2)$$

Note the first of these are the same coordinates as those for OP_1 and OP_3 as was predicted.

Testing the Spherical Coordinate Equations

Refer to the supplement *Spherical Coordinates* also

Say for $OP_1 = \langle x_1, y_1, z_1 \rangle = \langle -2,3,2 \rangle$

$$r = \rho = \sqrt{17}$$

$$lat = \phi = \cos^{-1}\left(\frac{hk}{\rho}\right) = \cos^{-1}\left(\frac{3}{\sqrt{17}}\right) = 0.7559$$

$$lon = \theta = \tan^{-1}\left(\frac{gk}{k}\right) = \tan^{-1}\left(\frac{-2}{2}\right) = -0.7853$$

Therefore the spherical coordinates are

$$(\rho, \theta, \phi) = (\sqrt{17}, -0.7853, 0.7559)$$

Reconverting to spherical coordinates using the equations given in the supplement:

$$z = \rho \sin\phi \cos\theta = \sqrt{17} \sin 0.7559 \cos -0.7853 = 2$$

$$x = \rho \sin\phi \sin\theta = \sqrt{17} \sin 0.7559 \sin -0.7853 = -2$$

$$y = \rho \cos\phi = \sqrt{17} \cos 0.7559 = 3$$

The spherical (3D polar) representation is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

Interesting points regarding distance on a sphere

Nautical mile

On a great circle, 1 minute of arc is one nautical mile.
The circumference of the earth is 360 * 60 = 21,600 nautical miles.
In meters, knowing that the mean radius of the earth is 6,370,000 meters, then one nautical mile is

$$nautical\_mile = \frac{2\pi * 6{,}370{,}000}{21{,}600} = 1{,}852.957889\,meters$$

This is just a little larger than the ordinary mile, which is

$$mile = 1{,}609.3\,meters$$

Spherical Equation for a Great Circle[11]

The following is for the ZYX axes.

Given two points (lat_1, lon_1), (lat_2, lon_2), the great circle that contains these two points is defined by points (lat_c, lon_c) that satisfy:

$$lat_c = \tan^{-1}\left(\frac{\sin(lat_1)*\cos(lat_2)*\sin(lon_c - lon_2) - \sin(lat_2)*\cos(lat_1)*\sin(lon_c - lon_1)}{\cos(lat_1)*\cos(lat_2)*\sin(lon_1 - lon_2)}\right)$$

The lateral angle extends over a range of 1 PI, while the longitudinal angle extends over 2 PI.

Let theta (lon_c) vary from 0 to 2π

I tried these spherical equations with negative results (though some wonderful swirls were produced)….

View the document "Kaplan-Hart_Bridges_2001.pdf" (in dome physics directory). On page 7, fig 9, the middle 'near miss' represents our geohut. Note the comment that it had been found in documents from Daniele Barbaro. *La Pratica Della Perspettiva*. 1569.

[11] copied from http://nsidc.org/data/tools/spheres/GCIntersect.html

Knowing arc length of the pentagon and circumference of the sphere, find the (vertical) proportion of the pentagon's location to its radius.

Given

Circumference: 10.250
Top Pentagon arc length: 0.42 // Buckminster: 0.424647373
Bottom pentagon arc length: (2*160) + (610 * 2) = 1.54

Calculations

radius

r = C / 2PI = 1.631338

Pentagon 1

$$\theta = \frac{arc\_length}{radius}$$

$$\theta = \frac{0.42}{1.630338} = 0.257457$$

$$length\_chord1 = \sqrt{2r^2(1-\cos\theta)} = \sqrt{2\left(\frac{C}{2\pi}\right)^2 * (1-\cos\theta)}$$

$$length\_chord1 = \sqrt{2\left(\frac{10.25}{2\pi}\right)^2 * (1-\cos(0.257))}$$

$$length\_chord1 = 0.418840985$$

Pentagon 2

$$\theta = \frac{1.54}{1.6313} = 0.94401028$$

$$length\_chord2 = \sqrt{2\left(\frac{10.25}{2\pi}\right)^2 * (1-\cos(0.944))}$$

$$length\_chord2 = 1.48345114$$

Radius of the circle circumscribing the pentagon

phi = 1.618034

$$side = radius\_pent\sqrt{3-phi}$$

$$radius\_pent = \frac{side}{\sqrt{3-phi}}$$

$$radius\_pent1 = \frac{0.418840985}{\sqrt{3-phi}} = 0.356287423$$

$$radius\_pent2 = \frac{1.48345114}{\sqrt{3-phi}} = 1.261898911$$

Ratio between the sphere's and pentagon's radii

$$ratio = \frac{pentagon\_radius}{sphere\_radius}$$

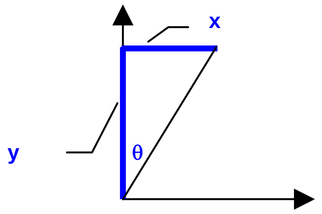
$$ratio\_pent1 = \frac{0.356287423}{1.631338} = 0.218401941$$

$$ratio\_pent2 = \frac{1.261886}{1.631338} = 0.773536068$$

Adjust to the parameters needed to instantiate the object

If for a sphere radius **r**, the pentagon radius is **x**, then this means that the cos of the angle θ in the figure is **x**. Therefore,

$$\theta = \cos^{-1} x$$



From which we can calculate y,

$$y = \sin\theta$$

For Pentagon 1: $\theta = \cos^{-1} ratio\_pent1 = 1.35061975$

$y = \sin\theta = 0.975858899$

For pentagon 2: $\theta = \cos^{-1} ratio\_pent2 = 0.686394442$

$y = \sin\theta = 0.633752279$

Calculations for Buckminster version[12]

For radius = 1:

PENTAGON 1

Arc of top pentagon = $\theta = 0.26030616$

$$length\_chord1 = \sqrt{2r^2(1-\cos\theta)} = 0.259571858 \quad \text{(X)}$$

$$radius\_pent1 = \frac{length\_chord1}{\sqrt{3-phi}} = 0.220805011 \quad \text{(Y)}$$

$$\theta = \cos^{-1} ratio\_pent1 = 1.34815655$$
$$y = \sin\theta = 0.975317972$$

PENTAGON 2

Arc of lower pentagon = $\theta = 0.94698656$

$$length\_chord2 = \sqrt{2r^2(1-\cos\theta)} = 0.911996026$$

$$radius\_pent2 = \frac{length\_chord2}{\sqrt{3-phi}} = 0.775790157$$

$$\theta = \cos^{-1} ratio\_pent2 = 0.682829945$$
$$y = \sin\theta = 0.630990991$$

[12] See Bamboo Dome illustration at back of this document

Graphically producing the great circles.

Refer to geohut.c in the geoHut project[13].

Assume the sphere's radius 1.

Earlier on we calculated the coordinates of the 2 pentagons positioned horizontally on the surface of a sphere. Every one of these points (with coordinates x, y, and z) can be represented as a vector with radius r pointing from the center of the sphere towards that point.

We then took 2 consecutive vectors (points) of one pentagon and found their common perpendicular vector by applying the cross product operation to them.
This perpendicular vector, let's call it P1, has its origin in the center of the sphere and has a length dependent on the angle between the 2 pentagon vectors.
P1 is the rotation axis around which we will rotate to produce a great circle.

(program: P1 = <xa, ya, za>)

We normalise the length of P1, effectively making its length 1.

Take one of the pentagon vectors, call it V1, and cross product it with P1. Normalise this third vector and call it P2.

Now we effectively created a right-handed 3D axis system.

(program: V1 = <xn, yn, zn>)
(program: P2 = <xn2, yn2, zn2>)

Let for i = x, y, z in turn…

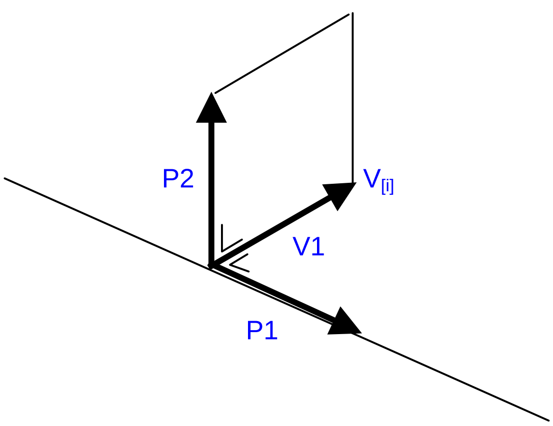
$V_{[i]}$ be the Pentagon's vector
$P1_{[i]}$ be the vector perpendicular to the 2 consecutive Pentagon vectors
$P2_{[i]}$ be the vector perpendicular to $V_{[i]}$ and $P1_{[i]}$

Then

For $0 < \theta < 2\pi$

Plot the Great Circle with

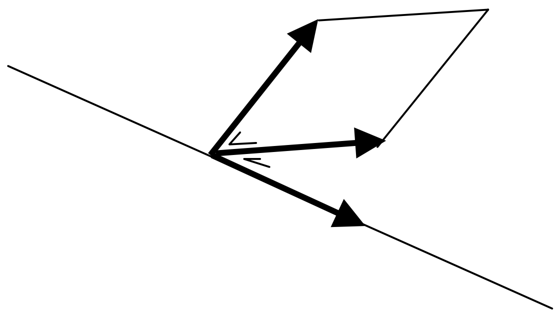
$$GC_{[i]} = V_{[i]} \cos\theta + P2_{[i]} \sin\theta$$



[13] Not publicly available as yet

Note that this equation is based on the equation for the circle

xcosθ+ ysin θ = r



Intersection of the great circle with the equator of the sphere

Building on this knowledge, we can derive the rotation angle for when the great circle cuts the Y axis. For then

$$GC_{[y]} = V_{[y]} \cos\theta + P2_{[y]} \sin\theta = 0$$

$$V_{[y]} \cos\theta = -P2_{[y]} \sin\theta$$

$$\theta = \tan^{-1}\left(\frac{-V_{[y]}}{P2_{[y]}}\right)$$

Feed this angle back into the initial formula

$$GC_{[i]} = V_{[i]} \cos\theta + P2_{[i]} \sin\theta$$

Intersection of plane with the cylinder

The cylinder is a less complex object than the sphere, and this is reflected in the mathematics… A simple combination of Cartesian with Parametric equations is all that's needed.

For the plane

$$aX + bY + cZ = 0 \qquad (1)$$

And the cylinder of radius 1,

$$Z^2 + X^2 = 1$$

The projection of the great circle on the ZX (horizontal) plane is represented by[14]

$$Z = \cos t \qquad X = \sin t \qquad 0 \le t \le 2\pi \qquad (2)$$

Derive Y from (1),

$$Y = \frac{-cZ - aX}{b}$$

Combining (1) with (2),

$$Z = \cos t \qquad X = \sin t \qquad Y = \frac{-c\cos t - a\sin t}{b} \qquad 0 \le t \le 2\pi$$

Interesting observation

It doesn't matter whether the cross product vector <aX, bY, cZ> was normalised or not. The results are the same.

For Y = 0, we already know the values of X and Z know from the great circle / equator intersection. In the program, the results are stored in xmid and ymid

So

$$Z = zmid = \cos t$$

$$X = xmid = \sin t$$

$$Y = \frac{-c * zmid - a * xmid}{b}$$

Therefore

$$t = a\cos zmid = a\sin xmid$$

[14] See "Calculus" 4th edition by James Stewart - ISBN: 0-534-35949-3 pp 873

Intersection of 2 curves on the cylinder

Equation for curve 1, for $0 \le t \le 2\pi$

$$Z = \cos t$$

$$X = \sin t$$

$$Y1 = \frac{-c\cos t - a\sin t}{b}$$

Equation for curve 2, for $0 \le t \le 2\pi$

$$Z = \cos t$$

$$X = \sin t$$

$$Y2 = \frac{-f\cos t - d\sin t}{e}$$

At point of intersection, Y1 = Y2

$$\frac{-c\cos t - a\sin t}{b} = \frac{-f\cos t - d\sin t}{e}$$

$$-ec\cos t - ea\sin t = -bf\cos t - bd\sin t$$

$$(bf - ec)\cos t = (ea - bd)\sin t$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{bf - ec}{ea - bd}$$

$$t = \tan^{-1}\left(\frac{bf - ec}{ea - bd}\right)$$

Feeding this value of t in any of the 2 equations will produce the coordinates of intersection.

Calculations needed to construct the dome

The remaining challenge is to give (Y) boundaries to both the spherical and cylindrical great circles and to divide each great circle section into equal parts so that the dome can be constructed from objects of equal size.
Also, we need to know the distances between points of intersection.

Calculate intersections to locate the lower rings.

The location of the upper and lower pentagons is optimised so that the whole of the dome can be constructed using 3 different lengths.

The sphere's great circles
We already determined where the great circles cut the equator.
Also, the great circle sections are subdivided equally in radians.

The cylinder's great ellipses
Subdivisions of the great ellipse sections are not equal.
Need to determine the intersections to find a suitable lower (Y) boundary for the base of the dome.

Defining a 3D curve in Polar coordinates

This section is not complete yet, though all the major equations are formalised.

Generally, for

$$a \le t \le b$$

And vector equation

$$r(t) = \langle f(t), g(t), h(t) \rangle$$

Or, equivalently, the parametric equation

$$x = f(t), y = g(t), z = h(t)$$

The arc length of the curve is

$$L = \int_b^a \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$L = \int_b^a \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

For the parametric equation

$$\mathbf{r}(t) = \sin t\mathbf{i} + \frac{-c\cos t - a\sin t}{b}\mathbf{j} + \cos t\mathbf{k}$$

Using the product rule,

$$\mathbf{r}'(t) = \cos t\mathbf{i} + \left[\frac{\frac{dy}{dt}(-\cos t - a\sin t)*b - (-\cos t - a\sin t)*\frac{dy}{dt}(b)}{b^2}\right]\mathbf{j} + (-\sin t)\mathbf{k}$$

$$\mathbf{r}'(t) = \cos t\mathbf{i} + \frac{c\sin t - a\cos t}{b}\mathbf{j} + (-\sin t)\mathbf{k}$$

$$|\mathbf{r}'(t)| = \sqrt{(\cos t)^2 + \left(\frac{c\sin t - a\cos t}{b}\right)^2 + (-\sin t)^2}$$

$$|\mathbf{r}'(t)| = \sqrt{(\cos t)^2 + \left(\frac{(c\sin t)^2 + 2(-a\cos t * c\sin t) + (-a\cos t)^2}{b^2}\right) + (-\sin t)^2}$$

Since $(\sin t)^2 + (\cos t)^2 = 1$

$$|\mathbf{r}'(t)| = \sqrt{1 + \left(\frac{c^2(\sin t)^2 - 2ac\cos t * \sin t + a^2(\cos t)^2}{b^2}\right)}$$

$$|\mathbf{r}'(t)| = \sqrt{1 + \frac{c^2}{b^2}(\sin t)^2 - \frac{2ac}{b^2}\cos t \sin t + \frac{a^2}{b^2}(\cos t)^2}$$

Convert to a polynomial form

let

$$u = 1 + \frac{a^2}{b^2}$$

$$v = 1 + \frac{c^2}{b^2}$$

$$w = -\frac{ac}{b^2}$$

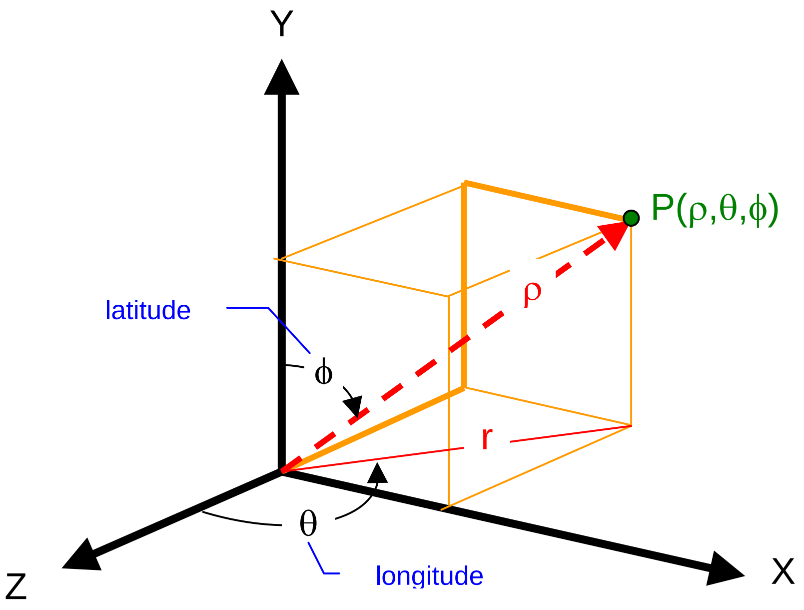
then

$$|\mathbf{r}'(t)| = \sqrt{u(\cos t)^2 + 2w\cos t \sin t + v(\sin t)^2}$$

Supplement - Spherical Coordinates

Standard representation of a point using spherical coordinates

Following diagram illustrates that a point in 3D space can be described by 2 angles. Note that some textbooks may interchange X, Y, and Z, but this does not alter the logic. The X, Y, Z configuration used here is the standard used in the OpenGL Graphics library and therefore relates to our discussion[15].



In the Y-r plane

$y = \rho \cos\phi$ (1)
$r = \rho \sin\phi$ (2)

In the Z-X plane

$z = r \cos\theta$ (3)
$x = r \sin\theta$ (4)

Spherical to rectangular

$z = \rho \sin\phi \cos\theta$ (2, 3)
$x = \rho \sin\phi \sin\theta$ (2, 4)
$y = \rho \cos\phi$ (1)

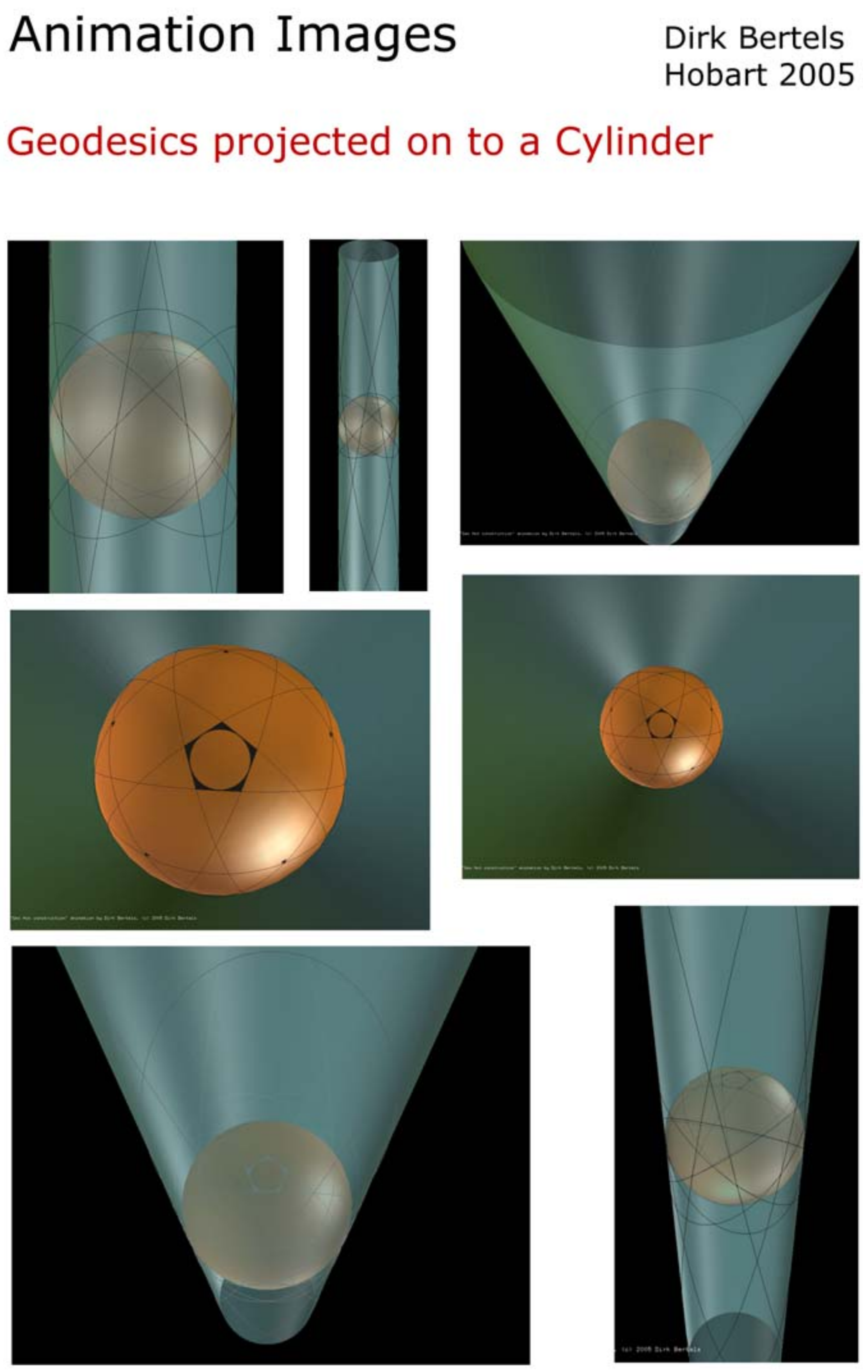
Rectangular to Spherical

$\phi = \cos^{-1} (y/\rho)$
$\theta = \tan^{-1} (x/z)$

[15] All equations have been checked and confirmed

Illustrations

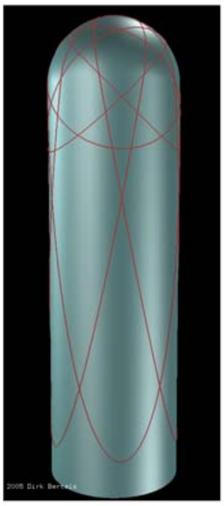
Following images are some still images taken from my C++ animation demonstrating the great circles on cylinders and domes.

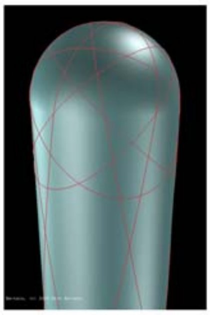


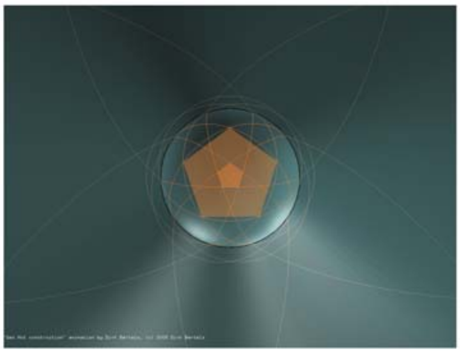
Animation Images

Dirk Bertels
Hobart 2005

Geodesics formed by 2 Pentagons projected on to a Cylinder

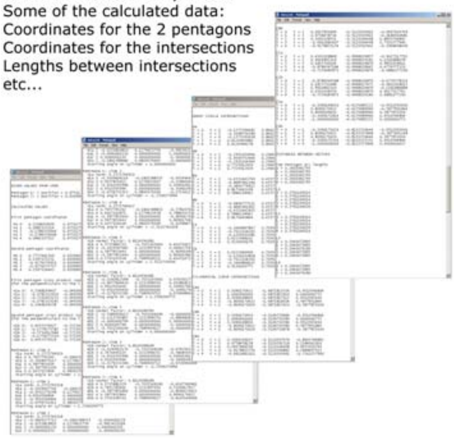






Print-outs

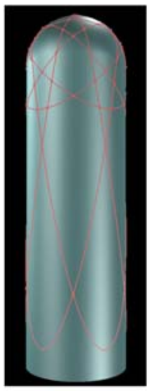
Version 5 produces print-outs based on the user's input data
Some of the calculated data:
Coordinates for the 2 pentagons
Coordinates for the intersections
Lengths between intersections
etc...

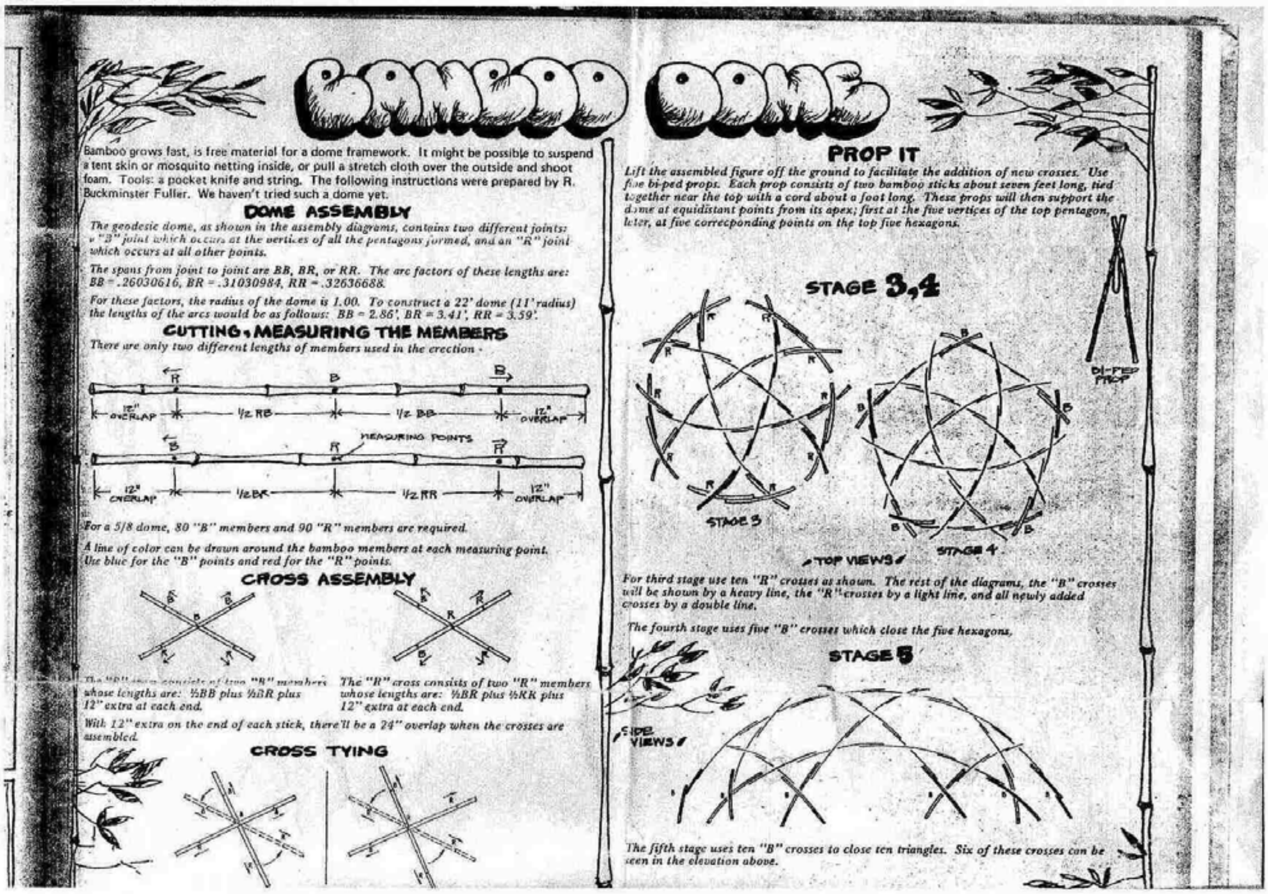


Intersections

Right:
Intersections for Pentagons located at the ratios determined by mr. Buckminster Fuller:
Pentagon 1: 0.9753179550
Pentagon 2: 0.6309909821

Below:
Intersections for the 2 Pentagons positioned at random locations





BAMBOO DOME

Bamboo grows fast, is free material for a dome framework. It might be possible to suspend a tent skin or mosquito netting inside, or pull a stretch cloth over the outside and shoot foam. Tools: a pocket knife and string. The following instructions were prepared by R. Buckminster Fuller. We haven't tried such a dome yet.

DOME ASSEMBLY

The geodesic dome, as shown in the assembly diagrams, contains two different joints: a "B" joint which occurs at the vertices of all the pentagons formed, and an "R" joint which occurs at all other points.

The spans from joint to joint are BB, BR, or RR. The arc factors of these lengths are: BB = .26030616, BR = .31030984, RR = .32636688.

For these factors, the radius of the dome is 1.00. To construct a 22' dome (11' radius) the lengths of the arcs would be as follows: BB = 2.86', BR = 3.41', RR = 3.59'.

CUTTING, MEASURING THE MEMBERS

There are only two different lengths of members used in the erection -

For a 5/8 dome, 80 "B" members and 90 "R" members are required.

A line of color can be drawn around the bamboo members at each measuring point. Use blue for the "B" points and red for the "R" points.

CROSS ASSEMBLY

The "B" cross consists of two "B" members whose lengths are: ½BB plus ½BR plus 12" extra at each end.

The "R" cross consists of two "R" members whose lengths are: ½BR plus ½RR plus 12" extra at each end.

With 12" extra on the end of each stick, there'll be a 24" overlap when the crosses are assembled.

CROSS TYING

PROP IT

Lift the assembled figure off the ground to facilitate the addition of new crosses. Use five bi-ped props. Each prop consists of two bamboo sticks about seven feet long, tied together near the top with a cord about a foot long. These props will then support the dome at equidistant points from its apex; first at the five vertices of the top pentagon, later, at five corresponding points on the top five hexagons.

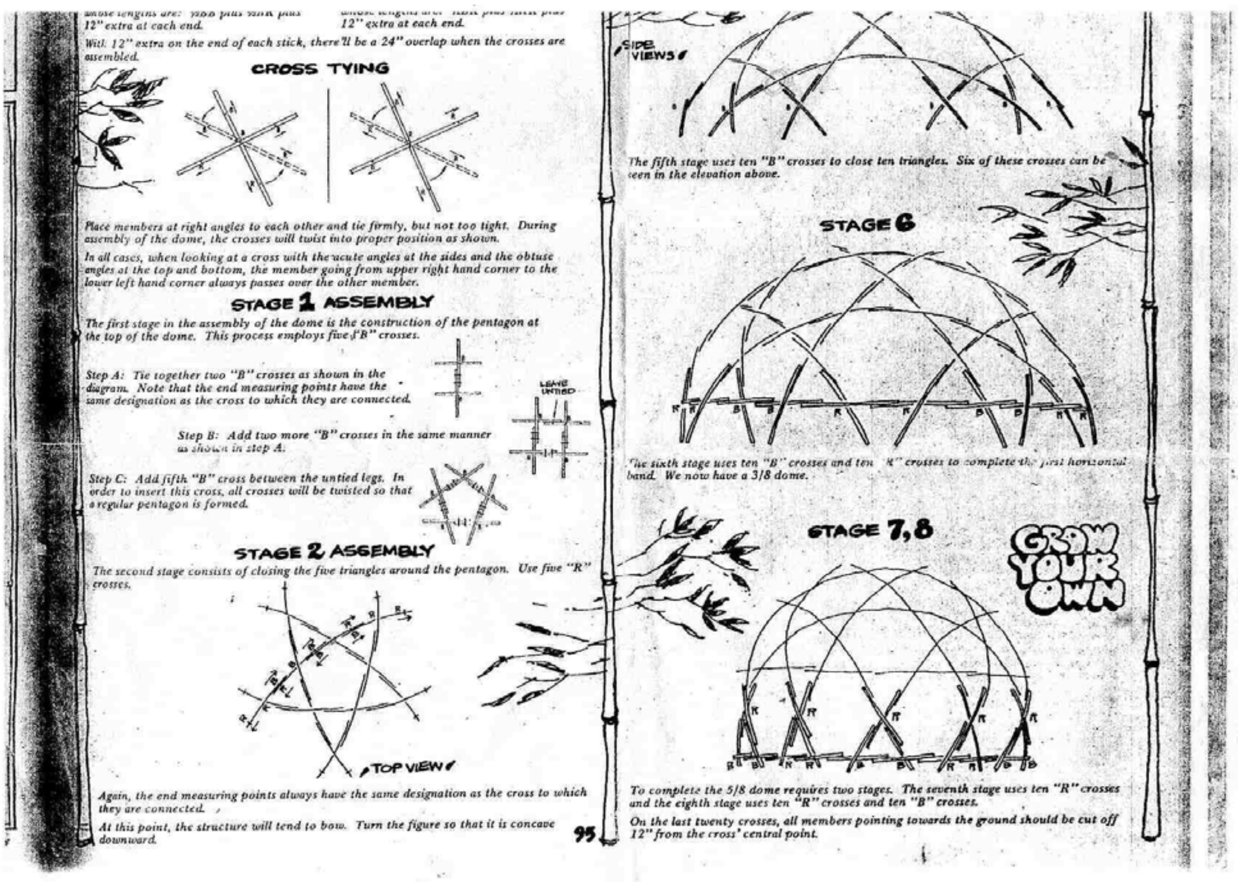
STAGE 3,4

For third stage use ten "R" crosses as shown. The rest of the diagrams, the "B" crosses will be shown by a heavy line, the "R" crosses by a light line, and all newly added crosses by a double line.

The fourth stage uses five "B" crosses which close the five hexagons.

STAGE 5

The fifth stage uses ten "B" crosses to close ten triangles. Six of these crosses can be seen in the elevation above.



whose lengths are: ½BB plus ½BR plus 12" extra at each end. *whose lengths are: ½BR plus ½RR plus 12" extra at each end.*

With 12" extra on the end of each stick, there'll be a 24" overlap when the crosses are assembled.

CROSS TYTING

Place members at right angles to each other and tie firmly, but not too tight. During assembly of the dome, the crosses will twist into proper position as shown.

In all cases, when looking at a cross with the acute angles at the sides and the obtuse angles at the top and bottom, the member going from upper right hand corner to the lower left hand corner always passes over the other member.

STAGE 1 ASSEMBLY

The first stage in the assembly of the dome is the construction of the pentagon at the top of the dome. This process employs five "B" crosses.

Step A: Tie together two "B" crosses as shown in the diagram. Note that the end measuring points have the same designation as the cross to which they are connected.

Step B: Add two more "B" crosses in the same manner as shown in step A.

Step C: Add fifth "B" cross between the untied legs. In order to insert this cross, all crosses will be twisted so that a regular pentagon is formed.

STAGE 2 ASSEMBLY

The second stage consists of closing the five triangles around the pentagon. Use five "R" crosses.

Again, the end measuring points always have the same designation as the cross to which they are connected.

At this point, the structure will tend to bow. Turn the figure so that it is concave downward.

95

SIDE VIEWS

The fifth stage uses ten "B" crosses to close ten triangles. Six of these crosses can be seen in the elevation above.

STAGE 6

The sixth stage uses ten "B" crosses and ten "R" crosses to complete the first horizontal band. We now have a 3/8 dome.

STAGE 7,8

To complete the 5/8 dome requires two stages. The seventh stage uses ten "R" crosses and the eighth stage uses ten "R" crosses and ten "B" crosses.

On the last twenty crosses, all members pointing towards the ground should be cut off 12" from the cross' central point.

Following figure was taken from http://www.boeing-727.com/Data/fly%20odds/distance.html . This is a good web site to explore because of its practical nature.