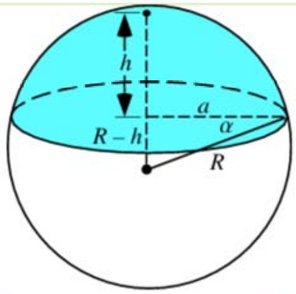
Spherical Cap



A spherical cap is the region of a sphere which lies above (or below) a given plane. If the plane passes through the center of the sphere, the cap is a called a hemisphere, and if the cap is cut by a second plane, the spherical frustum is called a spherical segment. However, Harris and Stocker (1998) use the term "spherical segment" as a synonym for what is here called a spherical cap and "zone" for spherical segment.

Let the sphere have radius R, then the volume of a spherical cap of height h and base radius a is given by the equation of a spherical segment

$$V_{\text{spherical segment}} = \tfrac{1}{6}\pi h\left(3a^2 + 3b^2 + h^2\right) \quad (1)$$

with $b = 0$, giving

$$V_{\text{cap}} = \tfrac{1}{6}\pi h\left(3a^2 + h^2\right). \quad (2)$$

Using the Pythagorean theorem gives

$$(R-h)^2 + a^2 = R^2, \quad (3)$$

which can be solved for a^2 as

$$a^2 = 2Rh - h^2, \quad (4)$$

so the radius of the base circle is

$$a = \sqrt{h(2R-h)}, \quad (5)$$

and plugging this in gives the equivalent formula

$$V_{\text{cap}} = \tfrac{1}{3}\pi h^2(3R-h). \quad (6)$$

In terms of the so-called contact angle (the angle between the normal to the sphere at the bottom of the cap and the base plane)

$$R - h = R\sin\alpha \quad (7)$$
$$\alpha \equiv \sin^{-1}\left(\frac{R-h}{R}\right), \quad (8)$$

so

$$V_{\text{cap}} = \tfrac{1}{3}\pi R^3\left(2 - 3\sin\alpha + \sin^3\alpha\right). \quad (9)$$

The geometric centroid occurs at a distance

$$\bar{z} = \frac{3(2R-h)^2}{4(3R-h)} \quad (10)$$

above the center of the sphere (Harris and Stocker 1998, p. 107).

The cap height h at which the spherical cap has volume equal to half a hemisphere is given by

$$h_{1/2} = 1 - 2\cos\left(\tfrac{4}{9}\pi\right). \quad (11)$$

Consider a cylindrical box enclosing the cap so that the top of the box is tangent to the top of the sphere. Then the enclosing box has volume

$$V_{\text{box}} = \pi a^2 h \quad (12)$$
$$= \pi(R\cos\alpha)^2[R(1-\sin\alpha)] \quad (13)$$
$$= \pi R^3\left(1 - \sin\alpha - \sin^2\alpha + \sin^3\alpha\right), \quad (14)$$

so the hollow volume between the cap and box is given by

$$V_{\text{box}} - V_{\text{cap}} = \tfrac{1}{3}\pi R^3\left(1 - 3\sin^2\alpha + 2\sin^3\alpha\right). \quad (15)$$

The surface area of the spherical cap is given by the same equation as for a general zone:

$$S_{\text{cap}} = 2\pi R h \quad (16)$$
$$= \pi\left(a^2 + h^2\right). \quad (17)$$

SEE ALSO:
Contact Angle, Dome, Frustum, Hemisphere, Solid of Revolution, Sphere, Spherical Ring, Spherical Segment, Spherical Wedge, Surface of Revolution, Torispherical Dome, Zone

REFERENCES:

Harris, J. W. and Stocker, H. "Spherical Segment (Spherical Cap)." §4.8.4 in *Handbook of Mathematics and Computational Science.* New York: Springer-Verlag, p. 107, 1998.

Kern, W. F. and Bland, J. R. "Spherical Segment." §36 in *Solid Mensuration with Proofs, 2nd ed.* New York: Wiley, pp. 97-102, 1948.

CITE THIS AS:

Weisstein, Eric W. "Spherical Cap." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/SphericalCap.html